ERG2011A
Tutorial 9

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Outline

- Fourier Series
  - Definition
  - Odd and Even Function
  - Half-range Expansion
- Fourier Transform
- Appendix

What is a Fourier series?

Let $f(x)$ be a PERIODIC function of period $2L$, and $f(x)$ satisfies some "commonly practical" conditions, then, $f(x)$ has a Fourier series given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x)$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \quad \text{and}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots \text{and}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots$$

where the a_0, a_n and b_n are the so-called Fourier coefficients

Odd and Even function

A function y = f(x) is *odd* if f(–x) = –f(x) (sinx)

- Graph is rotated at origin
- *and* $\int_{-L}^{L} f(x)dx = 0$

A function y = h(x) is *even* if h(–x) = h(x) (cosx)

- Graph is symmetric on y-axis
- *and* $\int_{-L}^{L} h(x)dx = 2\int_{0}^{L} h(x)dx$

Remarks:

f(x): *odd* and h(x): *even* $\Rightarrow$ p(x) = f(x)h(x): *odd*

Both f(x) and h(x): *odd* or *even* $\Rightarrow$ p(x) = f(x)h(x): *even*

Fourier series of Odd and Even function

The Fourier series of an *odd* function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with coefficient

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x dx, \quad n = 1, 2, \cdots$$

The Fourier series of an *even* function is given by

$$h(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

with coefficient

$$a_0 = \frac{1}{L}\int_0^L f(x) dx, \quad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x dx, \quad n = 1, 2, \cdots$$

Example of Fourier Series (Odd)

Consider $f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases}$ and $f(x+2\pi) = f(x)$.

Consider the Fourier series representation of

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}[\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx] = \frac{k}{2\pi}(\pi - \pi) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}[\int_{-\pi}^{0} -k\cos nxdx + \int_{0}^{\pi} k\cos nxdx] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}[\int_{-\pi}^{0} -k\sin nxdx + \int_{0}^{\pi} k\sin nxdx]$$

$$= \frac{k}{\pi}(\left.\frac{\cos nx}{n}\right]_{-\pi}^{0} - \left.\frac{\cos nx}{n}\right]_{0}^{\pi}) = \frac{k}{\pi n}[1-(-1)^n - (-1)^n + 1)] = \frac{2k}{\pi n}[1-(-1)^n]$$

Therefore, $b_n = \begin{cases} 0 & \text{for } n = 2, 4, 6, \ldots \\ \dfrac{4k}{n\pi} & \text{for } n = 1, 3, 5, \ldots \end{cases}$

Thus, $f(x) = \dfrac{4k}{\pi}(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \ldots)$

Fourier series of Half-range Expansion

Motivation

- f(x) is only defined on the interval [0, L]
- Can not be represented by Fourier series

Example:

Consider a function $f(x)$ which is only defined for the interval $0 \le x \le L$, then we can construct

$$f_1(x) = \begin{cases} f(x) \text{ for } 0 \le x \le L \\ -f(x) \text{ for } -L \le x \le 0 \end{cases} \text{ and } f_1(x+2L) = f_1(x)$$

Then, $f_1(x)$ is a periodic, ODD, function of period $2L$, and thus, we can obtain the Fourier series expansion for $f_1(x)$, i.e.

$$f_1(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

where $$b_n = \frac{1}{L}\int_{-L}^{L} f_1(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Once b_n's are determined, $f_1(x)$ can be used to represent $f(x)$ for $0 \le x \le L$ because $f(x) = f_1(x) =$ for $0 \le x \le L$.

Fourier Transform

Let f(x) be a given function, the Fourier transform of f(x) is given by

$$F(iw) = \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Conversely

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(iw) e^{iwt} dw$$

and is called Inverse F.T. operation.

Example:

Find the Fourier transform of $f(x) = \begin{cases} k, & \text{for } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

$$F(i\omega) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{0}^{a} k e^{-i\omega x} dx = \frac{k}{-i\omega}(e^{-i\omega a} - 1) = \frac{i \cdot k}{\omega}(e^{-i\omega a} - 1)$$

Note that $F(i\omega)$ is in general a complex-value function.

Appendix:

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

Appendix:

TABLE 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t - t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega-\omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega + \omega_0)]$
$\text{rect}(t/T)$	$T\text{sinc}\,(\omega T/2)$
$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$

Thanks☺