

ERG2011A
Tutorial 8

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Outline

- Laplace transform
 - Definition
 - “Shifting” theorem
 - L.T of some basic functions
- L.T of derivatives
 - Solving ODES

Definition of Laplace transform

Let $f(t)$ be a given function that is defined for $t \geq 0$

The Laplace transform of $f(t)$ is given by

$$F(s) = L(f) = \int_0^{\infty} e^{-st} f(t) dt$$

Conversely

$$f(t) = L^{-1}(F)$$

where $L^{-1}(\cdot)$ is the so-called Inverse L.T. operation.

Motivation:

- Solve a given “hard” problem indirectly
 - E.g. To solve: $y'' - y = t$, $y(0) = 1$, $y'(0) = 1$

$$[y = e^t + \sinh t - t]$$

The “Shifting” theorem for L.T

If $F(s) = \mathcal{L}(f(t))$ for $s > k$, then $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$ for $(s - a) > k$

Conversely, $e^{at} f(t) = \mathcal{L}^{-1}\{F(s - a)\} \dots\dots\dots (*)$

Proof:

$$F(s - a) = \int_0^{\infty} e^{-(s-a)t} f(t) dt = \int_0^{\infty} e^{-st} [e^{at} f(t)] dt = \mathcal{L}(e^{at} f(t)) \dots\dots\dots (**)$$

If $F(s) = \int_0^{\infty} e^{-st} f(t) dt$ exists (i.e. is finite) for $s > k$,

then the 1st integral in (***) exists for $(s - a) > k$

Taking the Inverse Transform of the both sides of (**), we get (*)

L.T. of some basic functions

	$f(t)$	$F(s)$		$f(t)$	$F(s)$
1	1	$1/s$	7	$\cos wt$	$s/(s^2 + w^2)$
2	t	$1/s^2$	8	$\sin wt$	$w/(s^2 + w^2)$
3	t^2	$2!/s^3$	9	$\cosh at$	$s/(s^2 - a^2)$
4	t^n $n = 0, 1, \dots$	$n!/s^{n+1}$	10	$\sinh at$	$a/(s^2 - a^2)$
5	t^a a: positive	$\Gamma(a+1)/s^{a+1}$	11	$e^{at} \cos wt$	$(s-a) / [(s - a)^2 + w^2]$
6	e^{at}	$1/(s-a)$	12	$e^{at} \sin wt$	$w / [(s - a)^2 + w^2]$

L.T of derivatives

If $f(t)$ is continuous for all $t \geq 0$ and $|f(t)| \leq Me^{kt}$ for some constants M and k for $s > k$, AND $f'(t)$ is piecewise continuous on every finite interval for $t \geq 0$, then

$$\mathcal{L}\{f'(t)\} \text{ exists and } \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \text{ when } s > k$$

By applying (*) to $f''(t)$, we have:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Similarly, it can be shown that:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Application: Solve ODES

- E.g. $y'' + 2y' + y = e^{-t}$, $y(0) = -1$, $y'(0) = 1$ [$y = (t^2/2 - 1)e^{-t}$]
- Comparison with the usual method
 - Which one is better?

Thanks😊