ERG2011A
Tutorial 6

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Outline

- Power series
  - Definition
  - Method for solving ODES
- Legendre’s equation
- Frobenius method

Power series

An infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots.$$

- a_0, a_1, a_2… are constants, called the coefficient of the series
- x_0 is also a constant, called the “center” of the series
- Example ($x_0 = 0$):

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- Radius of convergence $R = \lim_{x \to \infty} \left| a_m / a_{m+1} \right|$ provided these limits exist and are not zero (If they are infinite, then the original form converges only at the center x_0)

Power series method for solving ODES

Given: y’ - y = 0 ---------------------------------------(*)

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

Thus,

$$y' - y = (a_1 + 2a_2 x + 3a_3 x^2 + \ldots) - (a_0 + a_1 x + a_2 x^2 + \ldots) = 0$$

Grouping terms of x^m, we have:

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots = 0$$

which gives:

$$a_1 = a_0, \quad a_2 = a_1/2 = a_0/2!, \quad a_3 = a_2/3 = a_1/(2*3) = a_0/3!, \ldots$$

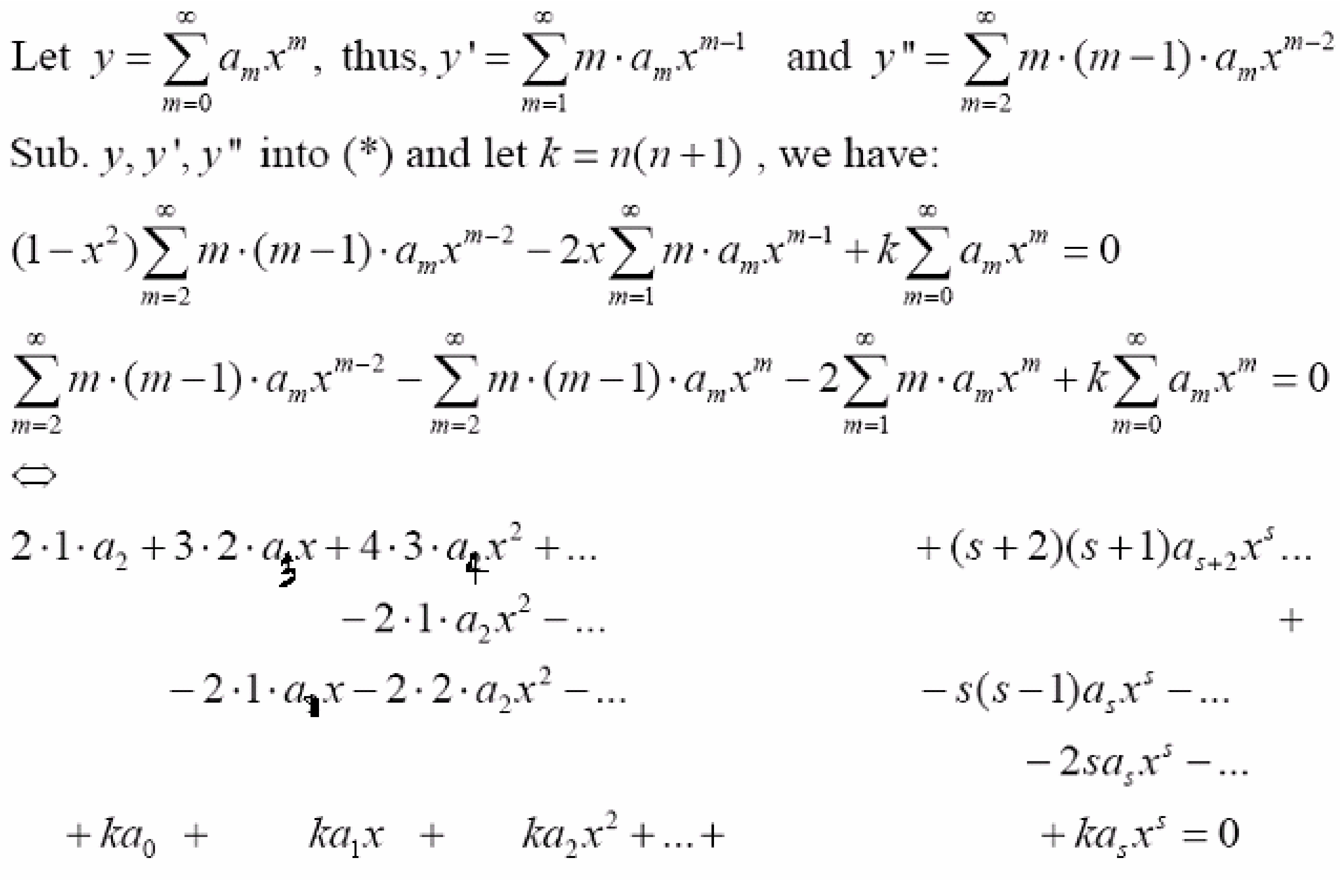
Thus,

$$y = a_0(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) = a_0 e^x$$ where a_0 is an arbitrary constant.

Legendre's equation

General form $(1 - x^2)y'' - 2xy' + n(n+1)\, y = 0$

Solution:



Let $y = \sum_{m=0}^{\infty} a_m x^m$, thus, $y' = \sum_{m=1}^{\infty} m \cdot a_m x^{m-1}$ and $y'' = \sum_{m=2}^{\infty} m \cdot (m-1) \cdot a_m x^{m-2}$

Sub. y, y', y'' into (*) and let $k = n(n+1)$, we have:

$$(1-x^2)\sum_{m=2}^{\infty} m \cdot (m-1) \cdot a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m \cdot a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=2}^{\infty} m \cdot (m-1) \cdot a_m x^{m-2} - \sum_{m=2}^{\infty} m \cdot (m-1) \cdot a_m x^m - 2\sum_{m=1}^{\infty} m \cdot a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$$

$\Leftrightarrow$

$$2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \ldots \qquad + (s+2)(s+1)a_{s+2}x^s \ldots$$

$$-2 \cdot 1 \cdot a_2 x^2 - \ldots \qquad +$$

$$-2 \cdot 1 \cdot a_1 x - 2 \cdot 2 \cdot a_2 x^2 - \ldots \qquad - s(s-1)a_s x^s - \ldots$$

$$-2sa_s x^s - \ldots$$

$$+ka_0 \; + \; ka_1 x \; + \; ka_2 x^2 + \ldots + \qquad + ka_s x^s = 0$$

Legendre's equation

- Comparing coefficients of each power of x, we thus have

$$2a_2+ka_0 = 0 \text{ (coefficient of } x^0\text{)}$$

$$6a_3 + [-2 + n(n+1)]a_1 = 0 \text{ (coefficient of } x^1\text{)}$$

- And in general,

$$a_s = -\frac{(s+1)(s+2)}{(n-s)(n+s+1)}a_{s+2}$$

- Successively,

$a_2 = -n(n+1)a_0/\ 2!$; $a_4 = -(n-2)(n+3)a_2/(4*3) = (n-2)\ n\ (n+1)(n+3)a_0/\ 4!$

And

$a_3 = -(n-1)(n+2)a_1/\ 3!$; $a_5 = -(n-3)(n+4)a_3/(5*4) = -(n-3)(n-1)(n+2)(n+4)a_1/5!$

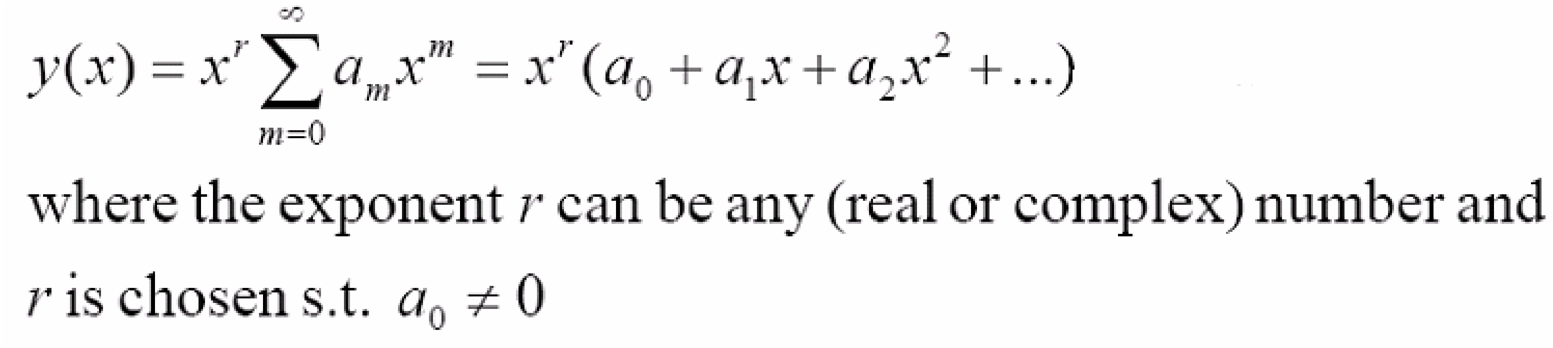
Or $y(x) = a_0\, y_1(x) + a_1\, y_2(x)$ where a_0, and a_1 are arbitrary constants and

$y_1(x) = 1 – n(n+1)x^2/2! + (n-2)n(n+1)(n+3)x^4/4! - \dots$

$y_2(x) = x – (n-1)(n+2)x^3/3! + (n-3)(n-1)(n+2)(n+4)x^5/5! - \dots$

Frobenius method

- Theorem: Any ODE of the form: y'' + b(x)y'/x +c(x)y/x^2 = 0 has at least one solution that can be expressed in the form of:



$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \ldots)$$

where the exponent r can be any (real or complex) number and

r is chosen s.t. $a_0 \neq 0$

Main question:

- How to do it?

Frobenius method

Multiple the standard form by x^2, we have:

$$x^2y'' + x\,b(x)y' + c(x)y = 0 \text{ ....(*)}$$

Express $b(x) = b_0 + b_1x + b_2x^2 + \ldots$; $c(x) = c_0 + c_1x + c_2x^2 + \ldots$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1x + a_2x^2 + \ldots) \text{............................(**)}$$

$$y'(x) = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \ldots]$$

$$y''(x) = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} = x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + (r+2)(r+1)a_2x^2 + \ldots]$$

Sub. y, y', y'' into (*), and equate the sum of coefficients of each power of x to zero, we have a system of equations with coeff. a_m as unknowns, in particular, consider the coeff. of $x^r = 0 \Rightarrow$

$$[r(r-1) + b_0r + c_0]a_0 = 0$$

since $a_0 \neq 0$ by construction of $y(x) \Rightarrow r(r-1) + b_0r + c_0 = 0$ ....(***)

(***) is the so-called Indicial Equation of the original ODE

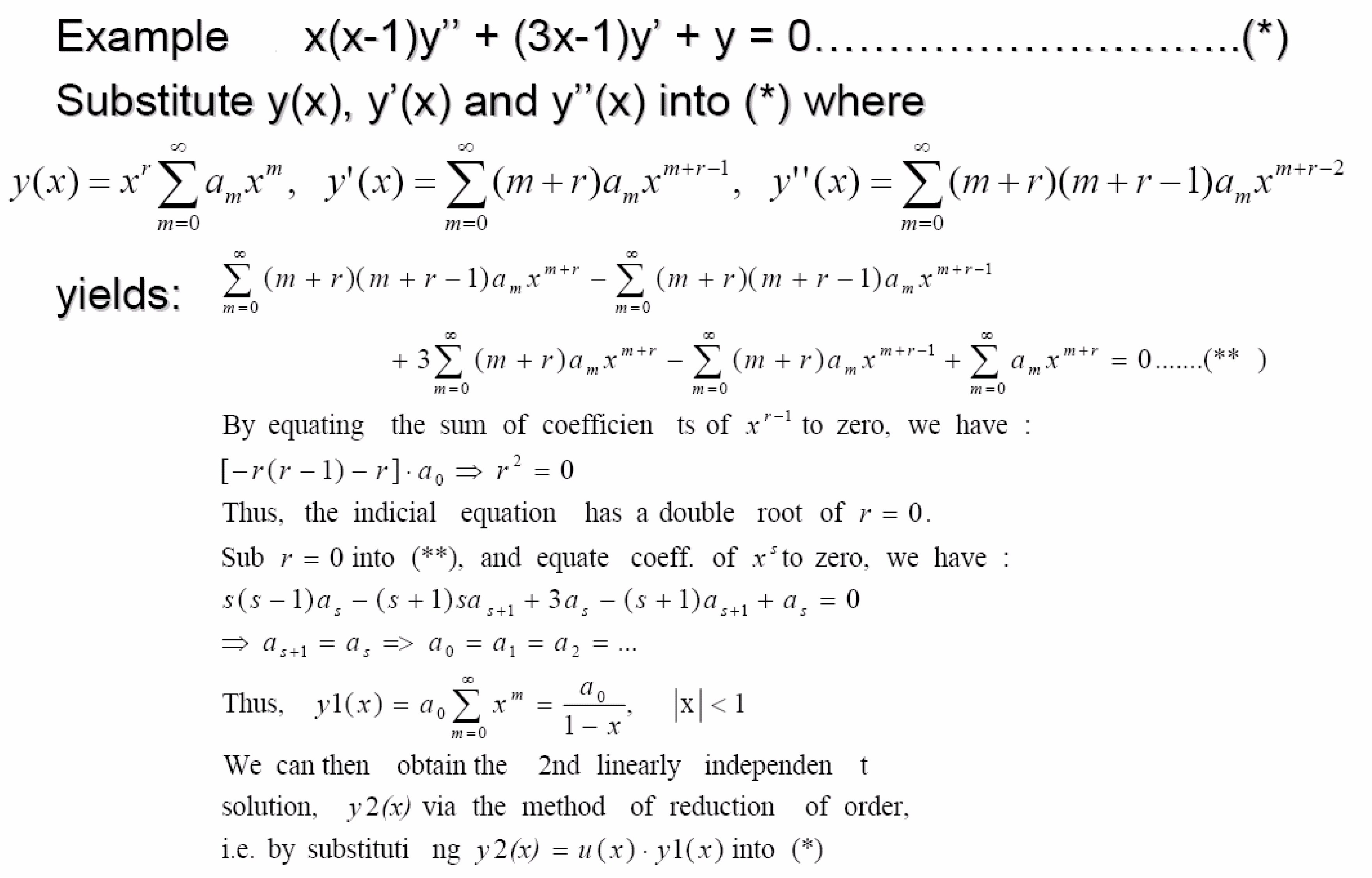
Frobenius method

- Case 1: Distinct roots r1 and r2 not differing by an integer 1, 2, 3, ····
  - $y_1(x) = x^{r1}(a_0 + a_1x + a_2x + \cdots)$
  - $y_2(x) = x^{r2}(A_0 + A_1x + A_2x + \cdots)$

- Case 2: A double root r1 = r2 = $(1 – b_0)/2$
  - $y_1(x) = x^{r1}(a_0 + a_1x + a_2x + \cdots)$
  - $y_2(x) = y_1(x)\ln x + x^{r1}(A_0 + A_1x + A_2x + \cdots)$

- Case 3: Roots r1 and r2 differing by an integer 1, 2, 3, ····
  - $y_1(x) = x^{r1}(a_0 + a_1x + a_2x + \cdots)$
  - $y_2(x) = ky_1(x)\ln x + x^{r2}(A_0 + A_1x + A_2x + \cdots)$

Frobenius method



Example x(x-1)y'' + (3x-1)y' + y = 0…………………………(*)

Substitute y(x), y'(x) and y''(x) into (*) where

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m, \quad y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}, \quad y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

yields:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1}$$

$$+ 3\sum_{m=0}^{\infty} (m+r) a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \text{.......(**)}$$

By equating the sum of coefficients of x^{r-1} to zero, we have :

$$[-r(r-1) - r] \cdot a_0 \Rightarrow r^2 = 0$$

Thus, the indicial equation has a double root of $r = 0$.

Sub $r = 0$ into (**), and equate coeff. of x^s to zero, we have :

$$s(s-1)a_s - (s+1)sa_{s+1} + 3a_s - (s+1)a_{s+1} + a_s = 0$$

$$\Rightarrow a_{s+1} = a_s \Rightarrow a_0 = a_1 = a_2 = \ldots$$

Thus, $y1(x) = a_0 \sum_{m=0}^{\infty} x^m = \dfrac{a_0}{1-x}, \quad |x| < 1$

We can then obtain the 2nd linearly independent solution, $y2(x)$ via the method of reduction of order, i.e. by substituting $y2(x) = u(x) \cdot y1(x)$ into (*)

Thank you☺