ERG2011A
Tutorial 5

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Outline

- Review

- Non-homogeneous second ODE
  - Undermined coefficients
  - Variation of parameters

Review (Quiz 3)

Consider the following differential equation

$$\theta'' + (L/g)\theta = 0,$$

(a) Find the general solution $\theta(t)$.

(b) Prove that regardless of initial conditions, find the time period of $\theta(t)$.

Solution:

- (a) can be solved by using characteristic equation. Observe that we can first obtain $\lambda^2 = -L/g$ by setting $\theta = e^{\lambda t}$, then the general solution is

$$\theta(t) = A\cos\left(\sqrt{\frac{L}{g}}\right)t + B\sin\left(\sqrt{\frac{L}{g}}\right)t$$

- (b) can be solved by solving $\theta(t) = \theta(t + T)$ and thus $T = 2\pi\sqrt{\frac{g}{L}}$

Non-homogeneous second ODE

- General form: $y'' + p(x)y' + q(x)y = r(x) \neq 0$

- General solution: Summation of y_h and y_p
  - y_h : General solution of homogeneous case
  - y_p : Particular solution which can be found by some methods

- Method of "Undermined coefficients" – solve y_p
  - With the condition p and q are constant.
  - If $r(x) = e^{ax}$, x^n, sinx and cosx, then the corresponding setting $y_p = Ae^{ax}$, $C_n x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0$ and Asinx + Bsinx.

Method of “Undermined coefficients

Example: To solve $y'' - 3y' + 2y = e^x$.

Solution:

- The general solution is $y_h = Ae^x + Be^{2x}$ which can be obtained by using characteristic equation $\lambda^2 - 3\lambda + 2 = 0$ (It has roots $\lambda = 1$ and $\lambda = 2$).

- We now try $y_p = Cxe^x$ and substitute the particular form into the original equation, we obtain $C = -1$.

- The general solution is $y = y_h + y_p = Ae^x + Be^{2x} - xe^x$

Method of "Variation of parameters"

- Unlike the "undetermined coefficient" method, this approach works even for non-constant coefficient p(x) and q(x).

- General form: y'' + p(x)y' + q(x)y = r(x)

- Let y_1 and y_2 form a basis of sol. to the homogeneous equation y'' + p(x)y' + q(x)y = 0 and W is the Wronskian of y_1 and y_2, then particular solution is

$$y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

Note $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

Method of "Variation of parameters"

Example : To find y_p for $y''+y=\sec x$..........................(1)

$y_1=\cos x$ and $y_2=\sin x$ form a basis of the homogeneous eq. corr. to (1)

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'=\cos x\cos x-\sin x\cdot(-\sin x)=1$$

$$y_p=-y_1\int\frac{y_2r}{W}dx+y_2\int\frac{y_1r}{W}dx=-\cos x\int\frac{\sin x\sec x}{1}dx+\sin x\int\frac{\cos x\sec x}{1}dx$$

$$=\cos x\ln\left|\cos x\right|+x\sin x$$

Therefore, the general solution y of (1) is given by :

$$y=y_h+y_p=c_1\cos x+c_2\sin x+\cos x\ln\left|\cos x\right|+x\sin x$$

$$=(c_1+\ln\left|\cos x\right|)\cos x+(c_2+x)\sin x$$

Thank you☺