ERG2011A
Tutorial 4

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Outline

- Review
  - Concept of linear independence
  - Reduction of order
- E–C equation
  - Two real roots
  - A real double root
  - Complex conjugate roots

Review (definition of linear independence)

- We call two functions $y_1(x)$ and $y_2(x)$ linearly independent if and only if the equation $ay_1(x) + by_2(x) = 0$ has only trivial solution (i.e. $a = b = 0$).

- Conversely, we say $y_1(x)$ and $y_2(x)$ are linearly dependent if $ay_1(x) + by_2(x) = 0$ for some constant a and b that are not both zero.
  - A simple strategy: Assume a is not zero, then we know $y_1(x) = -(b/a)\, y_2(x)$, which implies $y_1(x)$ and $y_2(x)$ are simply proportional to each other.

- Examples:
  - $y_1(x) = e^x$ and $y_2(x) = 2e^x$
  - $y_1(x) = \sin(x)$ and $y_2(x) = \cos(x)$

Review (Reduction of order)

- How to obtain a basis if one solution is known?
  - In the last tutorial, we showed the general procedure to solve it. (The key ingredient is to find $U = u^{'}(x) = y_1^{-2} \exp\left(-\int p(x)dx\right)$ and $y_2(x) = y_1(x)\, u\,(x)$)

- Find the general solution for the homogeneous linear equation $x^2y'' - xy' + y = 0$ with one solution $y_1(x) = x$.

  Solution: A solution is $y_1(x) = x$ because $y' = 1$ and $y'' = 0$, so that substitution gives $-x + x = 0$. Now comes an important point. Firstly, division by x^2 gives us the standard form

$$y'' - (1/x)\, y' + (1/x^2)y = 0$$

  Observe that $p(x) = -1/x$, $U = 1/x$ and $u(x) = \text{In}x$, which implies $y_2(x) = x\text{In}x$. By using superposition principle, we have the general solution is

$$y(x) = C_1 x + C_2\, x\text{In}x$$

  Note that x and xInx are linearly independent.

E–C equation

- A class of second order ODE that can be solved
  - Consider the equation $x^2y'' + axy' + by = 0$, where a and b are constant.
  - Assume the solution is of the form $y = x^m$, then $y' = mx^{m-1}$ and $y'' = m(m - 1)x^{m-2}$. By substitution, we obtain the auxiliary equation $m^2 + (a - 1)m + b = 0$. Let m1 and m2 be the roots of this equation, we can show the form of general solution is:
    - Two distinct real roots, $y = C_1x^{m1} + C_2x^{m2}$
    - A double root, $y = (C_1 + C_2 \text{In}\, x)x^{m1}$
    - Two complex roots, $y = x^r\left(A\cos\left(w\text{In}x\right) + B\sin\left(w\text{In}x\right)\right)$, where m1 = r + iw and m2 = r + iw

E–C equation

(Second solution of Euler-Cauchy equation with repeated roots). Consider the ODE

$$x^2y'' + axy' + by = 0.$$

As shown in class, x^m is a solution to this ODE, for m satisfying

$$m^2 + (a-1)m + b = 0. \tag{1}$$

If $(a-1)^2 \neq 4b$, then prove that there are two distinct values m_1 and m_2 that satisfy (??). On the other hand, if $(a-1)^2 = 4b$, use the method of reduction of order derived in class to prove that the two solutions equal $x^{(1-a)/2}$ and $(\ln(x))x^{(1-a)/2}$.

Thank you☺