

ERG2011A

Tutorial 11

Vincent Wang
Room 732, SHB

Gradient and Directional Derivative

- The gradient of some function f , denoted by “grad f ” or ∇f , is the vector:

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

- Differential operator: $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$
- For instance, if $f(x, y, z) = 2x + yz - 3y^2$, then grad $f = 2\mathbf{i} + (z - 6y)\mathbf{j} + y\mathbf{k}$
- The directional derivative of f at some point P in the direction of \mathbf{a} , denoted by $D_{\mathbf{a}}f$, is given by

$$D_{\mathbf{a}}f = \frac{\mathbf{a} \cdot \nabla f}{|\mathbf{a}|}$$

Example

- Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $P: (2, 1, 3)$ in the direction of the vector $a = i - 2k$.

Solution:

From the definition, we obtain $\text{grad } f = 4xi + 6yj + 2zk$

At the point $P(2, 1, 3)$, $\text{grad } f = 8i + 6j + 6k$

Then

$$D_a f = \frac{a \cdot \nabla f}{|a|} = \frac{(i - 2k) \cdot (8i + 6j + 6k)}{\sqrt{5}} = -\frac{4}{\sqrt{5}}$$

- *Remark:* Minus sign indicates that f decreases at P in the direction of a .

Divergence of a vector field

- Let \mathbf{v} be a differentiable vector function and V_1, V_2, V_3 be the components of \mathbf{v} . Then the divergence of \mathbf{v} , denoted by $\text{div } \mathbf{v}$ or $\nabla \cdot \mathbf{v}$, is given by

$$\nabla \cdot \mathbf{v} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

- $\nabla \cdot \mathbf{v}$ is a scalar, not a vector (different from vector ∇f)
- Prove $\text{div} (f \mathbf{v}) = f \text{div } \mathbf{v} + \mathbf{v} \cdot \nabla f$, where $f = e^{xyz}$ and $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$

Curl of a vector field

- Let $\mathbf{v} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be a differentiable vector function. Then the curl of \mathbf{v} is defined as

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \mathbf{k}$$

- For any twice continuously differentiable scalar function f , $\text{curl}(\text{grad } f) = 0$
- $\text{Curl}(f\mathbf{v}) = (\text{grad } f) \times \mathbf{v} + f \text{curl } \mathbf{v}$
- $\text{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl } \mathbf{u} - \mathbf{u} \cdot \text{curl } \mathbf{v}$

Good luck 😊