

ERG2011A

Tutorial 10

Vector Differential

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A. Length of a Curve

$$l = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \sqrt{\vec{r}'(t) \cdot \vec{r}'(t)} dt$$

You can imagine that $(x')^2 + (y')^2 + (z')^2$ is equivalent to the square of a line segment in the curve C, and we take integration to sum up the length of the line segment in [a, b].

See the hand drawing below:

Exercises:

$$\vec{r}(t) = a \cos t \vec{i} + a \sin t \vec{j} + ct \vec{k}, \quad \text{from } (a,0,0) \text{ to } (a,0,2\pi c)$$

(Set 8.5, Q28)

B. Gradient, Directional Derivatives, Divergence, Curl, Laplacian

By definition:

1) The vector derivative of a [scalar field](#) f is called the [gradient](#), and it can be represented as::

$$\nabla F = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]$$

It always points in the direction of greatest increase of f , and it has a [magnitude](#) equal to the maximum rate of increase at the point — just like a standard derivative. In particular, if a hill is defined as a height function over a plane $h(x,y)$, the 2d-gradient will be a vector in the xy-plane (sort of like an

arrow on a map) pointing in the steepest direction. The magnitude of the gradient is the slope of the hill in that steepest direction.

2) Directional Derivatives:

$$\nabla_{\vec{v}} F = \lim_{h \rightarrow 0} \frac{\Delta F}{h |\vec{v}|} = \lim_{h \rightarrow 0} \frac{\partial F}{\partial x} \frac{v_x}{|\vec{v}|} + \frac{\partial F}{\partial y} \frac{v_y}{|\vec{v}|} + \frac{\partial F}{\partial z} \frac{v_z}{|\vec{v}|}$$

It represent the rate of change of function F in the direction

of \vec{v} . Notice that it can be proved that $\nabla_{\vec{v}} F = \nabla F \cdot \frac{\vec{v}}{|\vec{v}|} = |\nabla F| \cos \theta$

which means that *directional*

derivatives is the projection of the gradient in direction \vec{v} .

Exercise:

Find the directional derivative of a two-dimensional scalar

field $f = x^2 + y^2$ at a point (1, 1) in the directions of the

following vectors:

(a) $\vec{b} = (1, 1)$, (b) $\vec{b} = (0, 1)$, and (c) $\vec{b} = (-1, 1)$.

3) Divergence:

The divergence of a vector field $\mathbf{v}(x,y,z) = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$

can be represented as:

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

The divergence is roughly a measure of a vector field's increase in the direction it points; but more accurately a measure of that field's tendency to converge on or repel from a point

Exercise:

Find $\nabla \cdot \mathbf{v}$ for the following vector fields:

$$\mathbf{v} = (f_1(y, z), f_2(x, z), f_3(x, y))$$

$$\mathbf{v} = (xf_1(y, z), yf_2(x, z), zf_3(x, y))$$

4) Curl:

The curl of a vector field $\mathbf{v}(x,y,z) = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$ can be

represented as:

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \times (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k})$$

or ,

$$\text{curl } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The curl at a point is proportional to the on-axis torque a tiny pinwheel would feel if it were centered at that point.

Exercise:

Find $\vec{n} \times \vec{v}$ for $\vec{v} = (y/x, z/y, x/z)$ at the points *(a) (x, y, z) and (b) $(-1, -1, -1)$.

Prove that $\vec{n} \times (f\vec{v}) = \vec{n} f \times \vec{v} + f\vec{n} \times \vec{v}$;

5) Laplacian:

The Laplace operator is a scalar operator that can be applied to scalar fields; it is defined as:

$$\nabla^2 F = \nabla \cdot \nabla F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

The Laplacian is ubiquitous throughout modern

mathematical physics, appearing in the Poisson's equation, the heat equation, the wave equation, and the Schrödinger equation — to name a few.

It should be note that F is scalar field and \mathbf{v} is vector field in the above notation. Also, $\text{grad } F$ has it physical meaning which represent the surface normal vector at point P , in other words, it is lies on the normal plane and perpendicular to tangent plane at P .

Exercises: $f = \cosh^2 x - \sinh^2 y$, find $\nabla^2 f$ (Set 8.10, Q18)