Tutorial 9 vector calculus

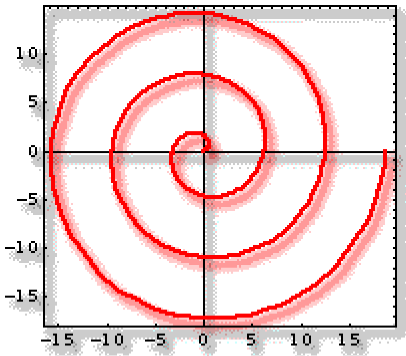
A：Basic concept for vector calculas

Parametric curves in 2D and 3D; arclength, curvature

A curve can be **parametrized** by a variable t (such as time), where its coords (two or three) are functions of t. Let's show one 2D curve and one 3D curve:

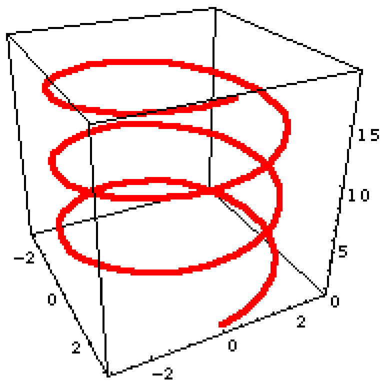
Here's a spiral curve in the xy-plane:

x = t cos(t) , y = t sin(t) ; 0 < t < 6 Pi .



Now here's a "helix" curve in 3-space:

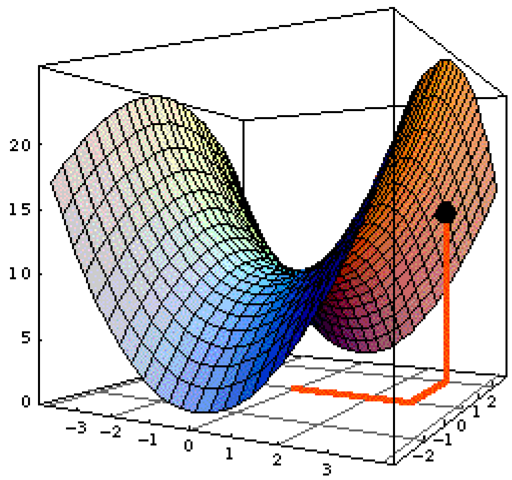
x = 3 cos(t) , y = 3 sin(t) , z = t ; 0 < t < 6 Pi .



The **arclength** is the distance an ant would crawl along the curve; it can be calculated by an appropriate integral hinging on the Pythagorean Theorem. The **curvature** is how fast the ant has to turn, in, say, degrees per centimeter.

Functions of two variables; surfaces

Think of z as a function of x and y ; **z = f(x,y)**.
For example we could have z = x^2 - y^2 + 10.
Then for each point (x,y) in the plane, we put a point (x, y, x^2 - y^2 + 10) in 3-space. One is (3, 2, 15). The collection of all these points forms a **surface**, which is the **graph** of z = f(x,y). Here's part of this surface:



B. Introduction of Vector Calculus

In this chapter, we consider function which depends on multi-variables instead of one variable, that is, $y=f(x_1,x_2,x_3,\ldots\ldots,x_n)$. We will consider the derivatives of this multi-variable functions. In particular, you have learn

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad \text{where } y=f(x).$$

Now, for the function z=f(x,y), we assume y=y1 which is fixed, then z=f(x,y1) can be written as z=f1(x) which depends on x only, so $\frac{dz}{dx}=\lim_{\Delta x\to 0}\frac{f1(x+\Delta x)-f1(x)}{\Delta x}$. That is, when we assume y is fixed, z=f(x,y) is just like a function which only depends on x, and we can find it derivatives w.r.t. x. So, we donate the operator $\frac{\partial}{\partial x}$ as the *Partial derivatives* of x, which means the rate of change of z in direction of x. In particular,

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

Similarily, $\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

For the function more than two variables, i.e. u=f(x,y,z) , we should write

$$\left(\frac{\partial u}{\partial y}\right)_{xz}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y}$$

which means that x and z are keep constant.

A. Total Differential of Function of Multi-variables

We are interested in finding Δz which is the small change of z where z=f(x,y). We switch back to the case y=f(x), where function only depends on one variable. We know that $\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$, so roughly speaking,

$$\Delta y \approx \frac{dy}{dx}\Delta x \quad .$$

If we try to generalize the properties to z=f(x,y), it can be shown that $\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$, by assuming both $\Delta x \to 0,\ \Delta y \to 0,$ then $\Delta z \to 0,$ so

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

. Notice that this is only the roughly explanation, the detaild proof is shown in lecture notes.

If we extend the above result to n-variables case, that is, there exists a function y=f(x1, x2, …….,xn), then

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

B.Mapping from n-dimension space to m-dimension space

Suppose we have a point X in the n-dimension space D^n, which the basis of D^n is (x_1, x_2,……,x_n), we want to find Y, the corresponding point of X in the m-dimension space E^m. Then we can set up a system of m equation with n variables, which f() is the mapping function.

$$\begin{cases} y_1 = f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ y_m = f_m(x_1, x_2, \ldots, x_n) \end{cases}$$ (Equation (1))

Also, we have

$$\begin{aligned} dy_1 &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \ldots + \frac{\partial f_1}{\partial x_n} dx_n \\ &\vdots \\ dy_m &= \frac{\partial f_m}{\partial x_1} dx_1 + \frac{\partial f_m}{\partial x_2} dx_2 + \ldots + \frac{\partial f_m}{\partial x_n} dx_n \end{aligned}$$

or

$$\begin{bmatrix} dy_1 \\ \vdots \\ dy_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}.$$

(Equation (2))

We define

$$\left(\frac{\partial f_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

which is called the Jacobian Matrix. Then we can find the corresponding point Y of X from D^n to E^m by equation (1), and also we can know the linear mapping with given points according to equation (2). In vector form, equation (1) can be written as $Y = (y_1, y_2, \ldots y_m) = F(X)$ ....(1), equation (2) can be written as $dY = F_X dX$..................................(2), where F is the vector function of f_i, F_X is the Jacobian Matrix function.