ERG2011A

Tutorial 8

Fourier Series & Fourier Transform

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A. Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \quad \text{and}$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots \text{and}$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

where f(x) is a function with the period T=2L

The series coefficient terms a_n, b_n are called the Fourier

coefficients.

Remainder: We can only discuss about Fourier series of a PERIODIC FUNCTION. In addition, Fourier coefficients must be convergence, so that the whole series converge to the function f(x).

B. Fourier series of Odd Function and Even Function

Let first review the properties of even and odd function:

A function f(x) is called EVEN FUNCTION if for any well-defined x in the function f(x),

f(x) = f(-x)

A function f(x) is called ODD FUNCTION if for any well-defined x in the function f(x),

f(x) = -f(x)

Graphically, even function is symmetric on y-axis, and odd function is rotated at origin.

For even function, integration has the following property:

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

For odd function, integration has the following property:

$$\int_{-L}^{L} f(x)\,dx = 0$$

Remarks: if f(x) is an odd function, g(x) is an even function, P(x)= f(x)g(x) is an odd function. If f(x), g(x) both are odd or both are even, P(x)=f(x)g(x) is an even function.

Observation: sin x is an odd function with period T=2 π . cos x is an even function with period T=2 π .

Therefore, with the properties of odd and even function, we can conclude that

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = 0$ if f(x) is an odd function,

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0$ if f(x) is an even function, for all n.

This is called the Fourier cosine series/Fourier sine series.

C. Half-range Expansion

For f(x) which is only define on the interval [0, L], we cannot simply represents it by Fourier series (since they are not exactly with period T=2L). Then we can use the following methods:

Let a function $g_1(x)$ such that $g_1(x)$ = f(x) for positive x, and $g_1(x)$ = f(-x) for negative x, and $g_1(x)$ is periodic with T=2L.

Then for even function $g_1(x)$, $b_n=0$ for all n,

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

Then for x in [0, L], we can use Fourier cosine series to represent f(x).

Similarly, we can define an odd function $g_2(x)$ such that it equals f(x) for positive x, and –f(x) for negative x. Then

$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$, the Fourier sine series represent f(x) in [0, L].

This is called the HALF-RANGE EXPANSION.

D. Complex Fourier series

Recall $\cos x = \frac{(e^{ix} + e^{-ix})}{2}$, $\sin x = \frac{(e^{ix} - e^{-ix})}{2i}$, we can rewrite the Fourier series in complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}dx$$

Compare the coefficient with the original expression, we have

$$c_n = \frac{1}{2}(a_n - ib_n)$$

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Exercises: Find the Fourier series expression of

1) $f(x) = x + |x|, \qquad -\pi < x < \pi$

2) $f(x) = \begin{cases} x^2 & \dfrac{-\pi}{2} < x < \dfrac{\pi}{2} \\ \dfrac{\pi^2}{4} & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$

.

E. Fourier Transform

$$F(i\omega) = \bullet \ \{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

This is called the FOURIER TRANSFORM.

$$f(x) = \bullet \ ^{-1}\{F(i\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega x}d\omega$$

This is called the INVERSE FOURIER TRANSFORM.

The Fourier Transform can use to represent the APERIODIC FUNCTION. However, it can also use to represent Periodic function as follows:

$$F(i\omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}\right\} = \sum_{n=-\infty}^{\infty} c_n \cdot 2\pi\delta(\omega - n\omega_0)$$

F. Compare Fourier Transform with Laplace Transform:

$F(s) = L(f) = \int_0^{\infty} e^{-st} f(t)dt$, we can see that Fourier transform is just a special case of Laplace transform, by limiting s=iω, which is a purely imaginary.

That is why Fourier transform has most properties similar to Laplace Transform:

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

then

TABLE 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t - t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega-\omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega + \omega_0)]$
$\text{rect}(t/T)$	$T\text{sinc}\ (\omega T/2)$
$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
$\text{rect}(t/T)\cos(\omega_0 t)$	$\frac{T}{2}\left[\text{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right) + \text{sinc}\left(\frac{(\omega + \omega_0)T}{2}\right)\right]$
$\frac{\beta}{\pi}\text{sinc}\ (\beta t)$	$\text{rect}(\omega/2\beta)$
$\text{tri}(t/T)$	$T\ \text{sinc}^2(T\ \omega/2)$
$\text{sinc}^2(Tt/2)$	$\frac{2\pi}{T}\text{tri}(\omega/T)$
$e^{-at}u(t), \text{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \text{Re}\{a\} > 0$	$\left(\frac{1}{a + j\omega}\right)^2$
$t^{n-1}e^{-at}u(t), \text{Re}\{a\} > 0$	$\frac{(n - 1)!}{(a + j\omega)^n}$
$e^{-a\|t\|}, \text{Re}\{a\} > 0$	$\frac{2a}{a^2 + \omega^2}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0)$	$\sum_{n=-\infty}^{\infty} \omega_0 G(n\omega_0)\delta(\omega-n\omega_0),\ \omega_0 = \frac{2\pi}{T_0}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{\infty} C_k\delta(\omega-k\omega_0),\ \omega_0 = \frac{2\pi}{T_0},\ C_k = \frac{1}{T_0}\int_{T_0} g(t)e^{-jk\omega_0 t}dt$
$\delta_T(t)$	$\sum_{k=-\infty}^{\infty} \omega_0\delta(\omega-k\omega_0)$