ERG2011A

Tutorial 7

Laplace Transform 2

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A. Definition

Laplace Transform:

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)dt$$

Inverse Laplace Transform:

$$f(t) = L^{-1}(F) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(s)e^{st} ds$$

Unit step function:

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

For example, the function y=|x| can be written as

y=x · u(t) - x · u(-t)

Delta function:

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

In other words, it just like an instantaneous impulse occurs at only one time t = a.

B. Properties of Laplace Transform

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

In more general case, the differentiation in time domain

should be $L(f^{(n)}(t))=s^nL(f(t))-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots\ldots-f^{(n-1)}(0)$.

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

C. Differential equation with variable coefficients

Differentiation in s-domain:

$$-tf(t) = L^{-1}\{F'(s)\}$$

Integration in s-domain:

$$\frac{f(t)}{t} = L^{-1}\{\int_s^{\infty} F(\tau)d\tau\}$$

We can use these properties to solve differential equation

with variable coefficients, for example:

$$(at+b)y''+(ct+d)y'+(et+f)y=0$$

Because

$$L(ty') = -\frac{d}{ds}Y'(s) = -\frac{d}{ds}[sY(s)-y(0)] = -Y(s)-sY'(s)$$

$$L(ty'') = -\frac{d}{ds}Y''(s) = -2sY(s)-s^2Y'(s)+y(0)$$

The equation reduces to a first order differential equation

Example: Solve $ty''+(1-t)y'+ny = 0$ (Refer to

lecture notes)

D. Convolution

We define the symbol (*) as convolution, and by definition:

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

Note that it is a function of t, τ is only a dummy variable. Some important properties also listed below:

$$f * g = g * f$$
$$f * (g1 + g2) = f * g1 + f * g2$$
$$(f * g) * v = f * (g * v)$$
$$\delta(t) * f(t) = f(t)$$

You must NOTICE THAT the convolution in the t-domain is equivalent to the multiplication in s-domain,

i.e. $f(t) * g(t) = F(s)G(s)$

where F(s) and G(s) is the Laplace Transform of f(t) and g(t) respectively.

Similarly, the multiplication in t-domain is equivalent to the convolution in s-domain, i.e.

$$g(t)f(t) = G(s)*F(s)$$

Exercises: $Y(s)=\dfrac{1}{(s+3)(s-2)}$, find its inverse fourier transform by convolution integral.

E. Systems of differential equation

We can use Laplace Transform helps to solve system of linear differential equation, for example,

$$y1' = -y1 + y2, \quad y2' = -y1 - y2, \qquad y1(0) = 1, \quad y2(0) = 0$$

Taking Laplace Transform on both sides of the equation yields:

$$sY1 - y1(0) = -Y1 + Y2 \Rightarrow sY1 - 1 = -Y1 + Y2 ------(1)$$

$$sY2 - y2(0) = -Y1 - Y2 \Rightarrow sY2 = -Y1 - Y2 ------(2)$$

solving (1) and (2), we get

$$Y1 = \frac{s+1}{(s+1)^2+1}, \qquad Y2 = \frac{-1}{(s+1)^2+1}$$

so $y1(t) = e^{-t}\cos t, \qquad y2(t) = -e^{-t}\sin t$

F. More Examples

1) $L(\sin(\omega t + \delta))$

2) $L^{-1}\left(\frac{1-7s}{(s-3)(s-1)(s+2)}\right)$

3) $L^{-1}\left(\frac{4}{s^2-2s-3}\right)$

4) $y''+ay'-2a^2y=0, \qquad y(0)=6, y'(0)=0$

5) $L^{-1}\left(\frac{1}{s^2+4s}\right)$

6) $L(t\sin\omega t)=\frac{2\omega s}{(s^2+\omega^2)^2}$, find

$L((2t-a)\sin(2\omega t-\omega a)$

7) $L^{-1}\left(\frac{s^2-\pi^2}{(s^2+\pi^2)^2}\right)$

8) Past midterm question:

Let $f(t)$ be a periodic function with period T, i.e.,

$$f(t+T)=f(t) \quad \text{for all } t \quad \text{and}$$

let $\quad g_1(t)=\begin{cases} f(t) & \text{for } 0\le t\le T \\ 0 & \text{otherwise}\end{cases}$

Show that $F(s)=\frac{G(s)}{1-e^{-sT}}$ where $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g_1(t)$ respectively.