Tutorial 5

Series Solution of Differential Equation

1. **radius of convergence**

$$R = 1 / \lim_{m\to\infty} \sqrt[m]{|a_m|} \qquad \text{or} \quad R = 1 / \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right|$$

So, why are we worried about the convergence of power series? Well in order for a series solution to a differential equation to exist at a particular *x* it will need to be convergent at that *x*. If it's not convergent at a given *x* then the series solution won't exist at that *x*. So, the convergence of power series is fairly important.

Practical Result: Practical Result: For an ODE: y'' + p(x)y'+q(x)y = r(x) …………..(*)
If p(x), q(x) and r(x) have power series representations, then (*) has
power series solutions.

2. Find the solution

2.1 Steps in power series method:

Step 1: Represent the coefficients of the equations $p(x)$ and $q(x)$ using power series at $x = x_0$ where $p(x)$ and $q(x)$ are analytic. Without loss of generality, we can assume $x_0 = 0$. Thus we assume

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Frequently, $p(x)$ and $q(x)$ are polynomials, so nothing we need to do.

Step 2: Assume that

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Then we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = a_2 + 6a_3 x + \cdots$$

Step 3: Substitute y, y' and y'' into the equation, then collect like powers of x and equate the sum of coefficient of each occurring power of x to zero (or, the coefficient of the same term on the right-hand side if $r(x)$ is not zero). This gives relations of coefficients.

Step 4: Solve for the coefficients.

2.2 index shift

Example 2 Write the following as a series that starts at *n=0* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1} (x+4)^{n+2}$$

Solution

An index shift is a fairly simple manipulation to perform. First we will notice that if we define *i=n-3* then when *n=3* we will have *i=0*. So what we'll do is rewrite the series in terms of *i* instead of *n*. We can do this by noting that *n=i+3*. So, everywhere we see an *n* in the actual series term we will replace it with an *i+3*. Doing this gives,

$$\begin{aligned}\sum_{n=3}^{\infty} n^2 a_{n-1} (x+4)^{n+2} &= \sum_{i=0}^{\infty} (i+3)^2 a_{i+3-1} (x+4)^{i+3+2} \\ &= \sum_{i=0}^{\infty} (i+3)^2 a_{i+2} (x+4)^{i+5}\end{aligned}$$

The upper limit won't change in this process since infinity minus three is still infinity.

The final step is to realize that the letter we use for the index doesn't matter and so we can just switch back to *n*'s.

$$\sum_{n=3}^{\infty} n^2 a_{n-1} (x+4)^{n+2} = \sum_{n=0}^{\infty} (n+3)^2 a_{n+2} (x+4)^{n+5}$$

2.3 simple example

Look at the equation: $y''+4y=0$

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

As in other techniques for solving differential equations, once we have a "guess" for the solutions, we plug it into the differential equation. Recall that

$$y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$

Plugging this information into the differential equation we obtain:

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + 4 \sum_{n=0}^{\infty} a_n t^n = 0.$$

Our next goal is to simplify this expression such that only one summation sign "$\sum$" remains. The obstacle we encounter is that the powers of both sums are different, t^{n-2} for the first sum and t^n for the second sum. We make them the same by shifting the index of the first sum by 2 units to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty} 4a_n t^n = 0.$$

Now we can combine the two sums as follows:

$$\sum_{n=0}^{\infty}\Big(\quad (n+2)(n+1)a_{n+2}t^n + 4a_n t^n \quad \Big) = 0,$$

and factor out t^n:

$$\sum_{n=0}^{\infty}\Big(\quad (n+2)(n+1)a_{n+2} + 4a_n \quad \Big) t^n = 0.$$

The power series on the left is identically equal to zero, consequently all of its coefficients are equal to 0:

$$(n+2)(n+1)a_{n+2} + 4a_n = 0 \text{ for all } n = 0, 1, 2, 3, \ldots$$

Solving these equations for the "highest index" $n+2$, we can rewrite as

$$a_{n+2} = -\frac{4}{(n+1)(n+2)}a_n \text{ for all } n = 0, 1, 2, 3, \ldots$$

Based on initial conditions: $y(0)=a_0$ $y'(0)=a_1$

$$a_2 = -\frac{4}{1 \cdot 2}a_0.$$

$$a_3 = -\frac{4}{2 \cdot 3}a_1.$$

$$a_4 = -\frac{4}{3 \cdot 4}a_2 = -\frac{4}{3 \cdot 4} \cdot \left(-\frac{4}{1 \cdot 2}a_0\right) = \frac{2^4}{4!}a_0.$$

$$a_5 = -\frac{4}{4\cdot 5}a_3 = -\frac{4}{4\cdot 5}\cdot\left(-\frac{4}{2\cdot 3}a_1\right) = \frac{2^5}{2\cdot 5!}a_1.$$

What do we know about the solutions to our differential equation at this point? They look like this:

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \ldots \\
&= a_0 + a_1 t - \frac{4}{1\cdot 2}a_0 t^2 - \frac{4}{2\cdot 3}a_1 t^3 + \ldots \\
&= a_0\left(1 - \frac{4}{1\cdot 2}t^2 + \frac{2^4}{4!}t^4 \ldots\right) \\
&\quad + a_1\left(t - \frac{4}{3!}t^3 + \frac{2^5}{2\cdot 5!}t^5 \ldots\right) \\
&= a_0\left(1 - \frac{2^2}{2!}t^2 + \frac{2^4}{4!}t^4 \ldots\right) \\
&\quad + \frac{a_1}{2}\left(2t - \frac{2^3}{3!}t^3 + \frac{2^5}{5!}t^5 \ldots\right) \\
&= a_0\left(1 - \frac{1}{2!}(2t)^2 + \frac{1}{4!}(2t)^4 \ldots\right) \\
&\quad + \frac{a_1}{2}\left((2t) - \frac{1}{3!}(2t)^3 + \frac{1}{5!}(2t)^5 \ldots\right)
\end{aligned}$$

Of course the power series inside the parentheses are the familiar functions cos(2t) and sin(2t):

$$y(t) = a_0\cos(2t) + \frac{a_1}{2}\sin(2t).$$

2.4 Legend’s equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m\cdot a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m \cdot (m-1) \cdot a_m x^{m-2}$$

Then,

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2}$$

So all the even term of a_i can be expressed in terms of a_0, and all the odd term of a_i can be expressed in terms of a_1.

Finally,

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

$$y_1(x) = 1 - n(n+1)x^2/2! + (n-2)n(n+1)(n+3)x^4/4! - \ldots$$

$$y_2(x) = x - (n-1)(n+2)x^3/3! + (n-3)(n-1)(n+2)(n+4)x^5/5! - \ldots$$

If n is even, then for a_n, a_{n+2}, a_{n+4}………..are all zero, $y_1(x)$ are polynomial of degree n. Similarily, for n is odd, $y_2(x)$ are polynomial of degree n.

With $a_n = \frac{(2n)!}{2^n (n!)^2}$, $a_0 = 1$, we can derive the *Legendre Polynomial* as follow:

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

This is very useful in many engineering analysis.

2.5 Frobenius method

The power series method (Theorem 6) requires that the coefficients of the differential equation *p*, *q* and *r* are analytic. However, several second-order differential equations of great importance, such as Bessel equation, have coefficients that are not analytic. If these differential equations have the form

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

then we can apply an extension to the power series method called the **Frobenius method**.

If b(x) and c(x) are analytic at x=0, then one of the solution is in the format

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \ldots)$$

The equation also has a second linearly independent solution that may be of similar form but with different *r* or may contain a logarithmic term.

In particular, we have the *indicial equation*, $r(r-1)+b_0r+c_0=0$

After solving, if r_1 and r_2 are:

Case 1 distinct roots not differing by an integer:

$$y_1(x) = x^{r1}(a_0+a_1x+a_2x^2+\ldots) \text{ and } y_2(x) = x^{r2}(A_0+A_1x+A_2x^2+\ldots)$$

Case 2 double roots:

$$y_1(x) = x^{r1}(a_0+a_1x+a_2x^2+\ldots) \text{ and}$$
$$y_2(x) = ky_1(x)\ \ln x + x^{r1}(A_0+A_1x+A_2x^2+\ldots)$$

Case 3 distinct roots differing by an integer

$$y_1(x) = x^{r1}(a_0+a_1x+a_2x^2+\ldots) \text{ and}$$
$$y_2(x) = ky_1(x)\ \ln x + x^{r2}(A_0+A_1x+A_2x^2+\ldots)$$

Example:

$xy''+(2x+1)y'+(x+1)y=0$ (set 5.4, Q9)

Solution:

Step 1: rewrite it as the standard form:

$$y'' + \frac{(1+2x)}{x}y' + \frac{(0+x+x^2)}{x^2}y = 0 \qquad (1)$$

Step 2: obtain the indicial equation r(r-1)+b_0r+c_0=0;

b_0=1, c_0=0;

The indicial equation is:

r(r-1)+r=0;

r1=r2=0

Step 3. get the first solution and the second solution

Therefore, one solution is:

$$y_1(x) = x^r \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^i$$

Substitute it to (1), we have:

$$\sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i + \frac{1+2x}{x}\sum_{i=0}^{\infty}(i+1)a_{i+1}x^i + \frac{x+1}{x}\sum_{i=0}^{\infty}a_i x^i = 0$$

$$\sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i + 2\sum_{i=0}^{\infty}(i+1)a_{i+1}x^i + \sum_{i=0}^{\infty}(i+1)a_{i+1}x^{i-1} + \sum_{i=0}^{\infty}a_i x^i + \sum_{i=0}^{\infty}a_i x^{i-1} = 0$$

$$\sum_{i=0}^{\infty}[(i+2)(i+1)a_{i+2} + 2(i+1)a_{i+1} + (i+2)a_{i+2} + a_i + a_{i+1}]x^i + a_1x^{-1} + a_0x^{-1} = 0$$

=>

$$a_1 = -a_0$$
$$a_{i+2} = -[\frac{2i+3}{(i+2)^2}a_{i+1} + \frac{1}{(i+2)^2}a_i]$$

=>

$$\begin{aligned} a_1 &= -a_0 \\ a_2 &= a_0/2 = a_0/2! \\ a_3 &= -a_0/6 = -a_0/3! \\ a_4 &= -a_0/24 = -a_0/4! \\ &\cdots \end{aligned}$$

Then the first solution is:

$$y_1(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 \sum_{i=0}^{\infty} \frac{(-x)^i}{i!} = a_0 e^{-x}$$

$$= e^{-x} \qquad (a_0 \triangleq 1)$$

To obtain the other solution based on the first one, we can suppose:

$y_2(x) = u(x) y_1(x)$ and

$$y_2'(x) = u'e^{-x} - ue^{-x}$$

$$y_2''(x) = u''e^{-x} - 2u'e^{-x} + ue^{-x}$$

Substitute them to (1), we obtain:

$$u''e^{-x} - 2u'e^{-x} + ue^{-x} + (2+1/x)(u'e^{-x} - ue^{-x}) + (1+1/x)(ue^{-x}) = 0$$

$$u''e^{-x} + u'e^{-x}/x = 0 \rightarrow \frac{du'}{dx} = -\frac{u'}{x} \rightarrow u' = 1/x \rightarrow u = \ln x$$

Therefore, the solution is:

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{-x} \ln x$$

About the homework

1. Where to add the constant