ERG2011A

Tutorial 3

2^{nd} Order Differential Equation

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A. Introduction And Properties of 2^{nd} Order

$$y'' + p(x)y' + q(x)y = r(x)$$

Homogeneous:

If r(x) = 0, it is called *Homogeneous*, else, it is *non-Homogeneous*.

Superposition:

Suppose there exists y_1 and y_2 that both satisfy the equation, then $C_1y_1+C_2y_2$ also satisfy the equation. This is called the *Superposition* properties of *Linear* 2^{nd} differential equation.

General Solution: For homogeneous 2^{nd} Order differential equation, the general solution is always in the format $C_1y_1+C_2y_2$, where C_1 and C_2 are arbitrary constants, y_1 and y_2 are *linear independent*.

B. Method Of Reduction Of Order

Situation: You know one solution y_1 for the equation, and you need to find another solution y_2.

We let a function of x, u(x), and let $y_2(x) = u(x)y_1(x)$. Then,

$$y_2' = u' y_1 + y_1' u$$

and

$$y_2'' = u''y_1 + 2u'y_1'+u* y_1''$$

Substitute into the original equation and then compute u(x).

Exercises: $x^2 y'' - 5xy' + 9y = 0, \qquad y_1 = x^3$

C. **Solve By Characteristics Equation**

We consider the homogeneous, constant coefficient format:

$$y'' + ay' + by = 0$$

Then we can assume the answer is in the format $e^{\lambda x}$.

We solve for the auxiliary equation $\lambda^2 + a\lambda + b = 0$ to get λ_1 and λ_2.

The general solution comes as follows:

Case One: two distinct real roots

The general solution is

$$y_h = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Case Two: two complex roots

The general solution is

$$y_h = e^{rx}(A\cos\omega x + B\sin\omega x)$$

where r is the real part of the roots, ω is the imaginary part of the roots.

Remarks:

$$y_h = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Now, λ_1, λ_2 is in the form $r \pm i\omega$, so

$$y_h = e^{rx}(C_1 e^{i\omega x} + C_2 e^{i\omega x})$$

for $e^{i\omega x} = \cos \omega x + i \sin \omega x$

$$y_h = e^{rx}(A \cos \omega x + B \sin \omega x)$$

Case 3: two double real roots

$$y_h = (C_1 + C_2 x)e^{\lambda x}$$

Exercises: $25y'' + 40y' + 16y = 0$ (Set 2.3, Q5)

D. Differential Operator

D is the operator which equivalent to $\frac{d}{dx}$. You can express the differential equation in terms of D:

P(D)[y] = (D^2 + D -6) y

P(D) can be factorize to simplify the equation.

E. Euler-Cauchy Equation

$$x^2 y'' + axy' + by = 0 \quad \text{................(**)}$$

We try the solution in the format $y=x^m$, then $y' = m*x^{m-1}$, $y'' = m*(m-1)*x^{m-2}$.

We can see that by substitution, the equation (**) will become $[m(m-1) + am + b]x^m = 0$

$$[m^2 + (a-1)m + b]x^m = 0$$

This is similar to the auxiliary equation with solution $y = e^{\lambda x}$. We solve and get m_1 and m_2 from the auxiliary equation, and then the general solution are:

Two distinct real roots: $y = C_1 x^{m1} + C_2 x^{m2}$

Two complex roots: $y = x^r$ (A cos[wlnx] + B sin [wlnx])

Two double roots: $y = [C1 + C2\ln x]x^{m1}$

Exercises: Proof the above result!! (*Hints :* $e^{m\ln x} = x^m$)

F. Non-homogenous 2nd Order

$$y'' + py' + qy = r(x)$$

If r(x) is not equal to zero, then it is a non-homogeneous equation, its solution is the summation of *General Solution* and *Particular Solution*:

$$y = y_h + y_p$$

y_h is the general solution which is the solution of homogeneous case. y_p is is the particular solution which are going to find by some following rules:

If r(x) =	Set g(x) =
e^{ax}	Ae^{ax}
x^n	$C_n x^n + C_{n-1} x^{n-1} + \ldots\ldots + C_1 x + C_0$
$\sin x / \cos x$	$A \sin x + B \cos x$

The constant are going to be found by comparing coefficients with r(x).

Exercises:

$y'' + 9y = r(x) \qquad r(x) = i)\,6\cos 3x \qquad ii)\,e^{2x} \qquad iii)\,8x$

Special case:

$y'' + 1.2y' + 0.36y = 4e^{-0.6x}, \qquad y(0) = 0, \qquad y'(0) = 1$