ERG2011A

Tutorial 2

Ordinary Differential Equation

Prepared by Derek Cheung (kschng5@ie.cuhk.edu.hk)

A. Finding Integrating Factor ($x^m y^n$)

Sometimes, instead of using the integrating formula, we can use inspection to work out the integrating factor in the format $x^m y^n$.

The concept is simple:

$P(x,y)\,dx + Q(x,y)\,dy = 0$

For $x^m y^n$ becomes the integrating factor and multiply to whole equation, then exact solution formed. So

$$\frac{\partial x^m y^n P}{\partial y} = \frac{\partial x^m y^n Q}{\partial x}$$

And then solve it.

Exercises $(a+1)ydx+(b+1)xdy=0,$ $x^a y^b$

B. Finding Integrating Factor (special format)

$$y' + p(x)y = r(x)$$

If r(x) = 0, it is said to be homogeneous, and the equation become separable. If no, it is non-homogeneous and need to solve by following:

Finding the integrating factor by $F(x) = e^{\int pdx}$

Then multiply the F(x) on both sides and solve it by

$$\frac{d(e^{\int pdx} y)}{dx} = e^{\int pdx} r(x)$$

Exercises:

$$y' + y\sin x = e^{\cos x}$$

In general, the format should be $y' + p(x)y = r(x)y^a$

which is non-linear.

Then you should transform it into

$$y^{-a}y' + p(x)y^{1-a} = r(x)$$

Setting $u = y^{1-a}$, $u' = (1-a)y^{-a}y'$,

Then $u' + (1-a)p(x)u = (1-a)r(x)$

Now it reduces to linear form and can be solved.

Exercises: $y'+\frac{1}{3}y=\frac{1}{3}(1-2x)y^4$

C. Finding Integrating Factor (Some special cases)

$$y'^2 - xy' + y = 0$$

Let p = y’, then differentiate the whole equation r.w.t. x:

$$\frac{d(p^2)}{dp} \bullet \frac{dp}{dx} - y' - xy'' + y' = 0$$

$$2pp' - xp' = 0$$

$$p' = 0 \qquad or \qquad p = \frac{x}{2}$$

$$y'' = 0 \qquad or \qquad y' = \frac{x}{2}$$

$$y = cx \qquad or \qquad y = \frac{x^2}{4} + C$$

D. Picard Iteration

-- derived from Existence Theorem and Uniqueness Theorem

$$y' = f(x, y) \qquad and \qquad y(x_0) = y_0$$

For this equation, we find the solution by Picard Iteration:

$$y(x) = y_0 + \int_{x_0}^{x} (t, y(t))\, dt$$

This is the equation that use for iteration, we start by y_0, and

then find y_1, and so on.

In general,

$$y_n(x) = y_0 + \int_{x_0}^{x} (t, y_{n-1}(t))\, dt$$

Exercises: $y' = x + y, \qquad y(0) = -1$