Q1. [25 marks] Solve $\frac{xy+1}{y}dx + \frac{2y-x}{y^2}dy = 0, \quad y \neq 0$

solution:

(1) $P(x,y) = (xy+1)/y$.

$Q(x,y) = (2y-x)/y^2$

$$\frac{\partial P}{\partial y} = -\frac{1}{y^2} \qquad \frac{\partial Q}{\partial x} = -\frac{1}{y^2}$$

Hence the equation is exact.

(2). $$f(x,y) = \int \frac{xy+1}{y}\,dx = \frac{1}{2}x^2 + \frac{x}{y} + k(y)$$

$$Q = \frac{\partial f}{\partial y} = \frac{-x}{y^2} + k'(y) = \frac{2y-x}{y^2}$$

$$\Rightarrow \quad k(y) = \int \frac{2}{y}\,dy = 2\log|y| + C$$

$\therefore\ f(x,y) = \frac{x^2}{2} + 2\log|y| + \frac{x}{y} = C$ is the solution

Q2.[25 marks]

a) Test the exactness of the following differential equation:

$(e^x - \sin y)dx + (\cos y)dy = 0$ ....(*)

b) With the help of the findings from part a), solve the differential equation (*)

Solution:

(a) $P(x, y) = e^x - \sin y.$

$Q(x, y) = \cos y$

$$\frac{\partial P}{\partial y} = -\cos y \neq \frac{\partial Q}{\partial x} = 0$$

$\therefore$ The equation is not exact

(b) Suppose $F_1(x)$ is the integrating factor, then:

$$\frac{1}{F_1(x)} \frac{dF_1(x)}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -1$$

$$\therefore F_1(x) = e^{-x}$$

$$U(x, y) = \int (e^x - \sin y) e^{-x} \, dx$$

$$= x + e^{-x}\sin y + k(y)$$

$$\frac{\partial U}{\partial y} = e^{-x}\cos y + k'(y) = Q(x, y) \cdot F_1(x) = e^{-x}\cos y$$

$$\therefore k(y) = c$$

The solution is:

$$U(x, y) = x + e^{-x}\sin y = C$$

Q3. [25 marks] Find the general and particular solution of the following differential equation:

$y''+4y'+4y=4x^2+6e^x$.........(**)

Solution :

(1) Find the general solution.

$y''+4y'+4y=0$

This is a constant coefficients equation.

suppose $y=e^{rx}$, then.

$r^2+4r+4=0 \Rightarrow r_1=r_2=-2$

$\therefore y_1(x)=e^{-2x} \quad y_2(x)=xe^{-2x}$

And the general solution: $y=C_1y_1(x)+C_2y_2(x)$

$=C_1e^{-2x}+C_2xe^{-2x}$.

(2) Find the particular solution.

Try $y_p(x)=C_0+C_1x+C_2x^2+C_3e^x$.

$\therefore y_p'(x)=C_1+2C_2x+C_3e^x$

$y_p''(x)=2C_2+C_3e^x$.

$\therefore \begin{cases} 4C_0+4C_1+2C_2=0 \\ 4C_1+8C_2=0. \\ 4C_2=4 \\ 4C_3+4C_3+C_3=6 \end{cases}$

$\Rightarrow C_0=\frac{3}{2} \quad C_1=-2 \quad C_2=1 \quad C_3=\frac{2}{3}$.

$\therefore y_p(x)=\frac{3}{2}+(-2)x+x^2+\frac{2}{3}e^x$

Q4. [25 marks]

Using the power series method to find one solution in the form of $y = x^r \sum_{i=0}^{\infty} a_i x^i$

for the following ODE: $x^2 y'' - (x + x^2) y' + y = 0$.

Solution :

$$y = x^r \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^{i+r}$$

substitute it to the equation.

$$\sum_0^{\infty} (i+r)(i+r-1) a_i x^{i+r} - \sum_0^{\infty} (i+r) a_i x^{i+r} - \sum_0^{\infty} (i+r) a_i x^{i+r+1} + \sum_0^{\infty} a_i x^{i+r} = 0$$

Define $a_{-1} = 0$

$$\therefore \sum_0^{\infty} \left\{ \left[(i+r)(i+r-1) - (i+r) + 1 \right] a_i - (n+r-1) a_{n-1} \right\} x^{n+r} = 0$$

obtain the indicial equation :

$$r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$$

Based on $r = 1$, we can obtain :

$$a_1 = a_0 \quad a_2 = \frac{1}{2} a_1 \quad a_3 = \frac{1}{3} a_2 \quad a_4 = \frac{1}{4} a_3 \cdots$$

$$\therefore y_1(x) = x\left(a_0 + \frac{a_0}{1!} x + \frac{a_0}{2!} x^2 + \cdots \right)$$

$$= a_0 x e^x = x e^x \text{ when } a_0 = 1$$

Suppose $y_2(x) = u(x) y_1(x) = x \cdot e^x \cdot u$, subsititute it to the equation.

$$x^2 (u'' y_1 + 2u' y_1' + u y_1'') - (x + x^2)(u' y_1 + u y_1') + u y_1 = 0$$

$$\Rightarrow x^3 u'' + x^3 u' + x^2 u' = 0$$

$$\Rightarrow u = \int \frac{1}{x} e^{-x} dx$$

$$\therefore y_2(x) = u(x) y_1(x) = \int \frac{1}{x} e^{-x} dx \cdot x e^x$$

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