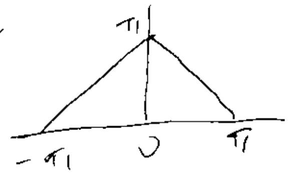
(11.1), Q15.



$$f(x) = \begin{cases} (x+\pi) & -\pi \le x \le 0 \\ (-x+\pi) & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Let $\{a_n\}$ be the fourier cosine series, $\{b_n\}$ be the fourier sine series.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} x+\pi \,dx + \frac{1}{2\pi}\int_{0}^{\pi} -x+\pi \,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}+\pi x\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{x^2}{2}+\pi x\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left(-\frac{\pi^2}{2}+\pi^2-\frac{\pi^2}{2}+\pi^2\right) = -\frac{\pi}{2}+\pi = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(-x+\pi)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\,d\left(\frac{\sin nx}{n}\right) + \frac{1}{\pi}\int_{0}^{\pi}(-x+\pi)\,d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{(x+\pi)\sin nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx + \left[\frac{(-x+\pi)\sin nx}{n}\right]_{0}^{\pi} + \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\cos nx}{n^2}\right]_{-\pi}^{0} - \left[\frac{\cos nx}{n^2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^n}{n^2} - \frac{(-1)^n-1}{n^2}\right) = \frac{2}{\pi n^2}\left(1-(-1)^n\right)$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{4}{\pi n^2} & \text{when } n \text{ is odd} \end{cases}$$

$b_n = 0$ for all n since $f(x)$ is even.

So $$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

(1, 2),

Q8, $f(x) = \begin{cases} 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1, \end{cases} \quad p = 2$

$f(x)$ is an even function, so $b_n = 0$ for all n.

$$a_0 = \frac{1}{2}\int_{-1}^{0} 1+x \, dx + \frac{1}{2}\int_0^1 1-x \, dx$$

$$= \frac{1}{2}\left[x + \frac{x^2}{2}\right]_{-1}^{0} + \frac{1}{2}\left[x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + 1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx$$

$$= \int_{-1}^{0} ((1+x)) \cos n\pi x \, dx + \int_0^1 ((1-x)) \cos n\pi x \, dx$$

$$= \left[\frac{((1+x)) \sin n\pi x}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0} \frac{\sin n\pi x}{n\pi} dx + \left[\frac{((1-x)) \sin n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\sin n\pi x}{n\pi} dx$$

$$= \frac{2}{n^2\pi^2}\left(1 - (-1)^n\right)$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{4}{\pi^2 n^2} & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$$

Q14, $f_1(x) = 1 + |x|, \quad f_2(x) = |x|, \quad p = 2$

$$\frac{1}{2}\int_{-1}^{1} 1 \, dx = \frac{1}{2} 2 = 1$$

$$\int_{-1}^{1} \cos n\pi x \, dx = \left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} = 0$$

$$\therefore f(x) = -\frac{1}{2} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right)$$

(1, 3)

Q (1), In fact, $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

which is same as the first question.

So $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

(2),

$f(x) = \begin{cases} -2x^2 & -1 < x < 0 \\ 2x^2 & 0 < x < 1 \end{cases}$

Since $f(x)$ is odd function, $a_n = 0$ for all n.

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x) \sin n\pi x \, dx \\
&= \int_{-1}^{0} -2x^2 \sin n\pi x \, dx + \int_0^1 2x^2 \sin n\pi x \, dx \\
&= \left[\frac{2x^2 \cos n\pi x}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0} \frac{4x \cos n\pi x}{n\pi} dx \\
&\quad + \left[\frac{-2x^2 \cos n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{4x \cos n\pi x}{n\pi} dx \\
&= \frac{2(-1)^{n+1}}{n\pi} + \frac{2(-1)^{n+1}}{n\pi} + \int_{-1}^{0} \frac{4 \sin n\pi x}{n^2\pi^2} dx - \int_0^1 \frac{4 \sin n\pi x}{n^2\pi^2} dx \\
&\cancel{= \frac{4(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}\left[-\cos n\pi x\right]_{-1}^{1}} \\
&\cancel{= \frac{4}{n\pi}(-1)^{n+1} = \begin{cases} \frac{4}{n\pi} & \text{when } n \text{ is odd} \\ -\frac{4}{n\pi} & \text{when } n \text{ is even} \end{cases}} \\
&= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}\left(\left[-\cos n\pi x\right]_{-1}^{0} + \left[\cos n\pi x\right]_0^1\right) \\
&= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}\left(-1 + (-1)^n + (-1)^n - 1\right) \\
&= \frac{4}{n\pi}(-1)^{n+1} + \frac{8}{n^3\pi^3}\left((-1)^n - 1\right) \\
&= \begin{cases} -\frac{4}{n\pi} & \text{when } n \text{ is even} \\ \frac{4}{n^3\pi^3}(n^2\pi^2 - 4) & \text{when } n \text{ is odd} \end{cases}
\end{aligned}
$$

$\therefore f(x) = \frac{4}{\pi^3}\left((\pi^2 - 4)\sin \pi x + \frac{1}{27}(9\pi^2 - 4)\sin 3\pi x + \cdots\right)$

$\quad - \frac{2}{\pi}\left(\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \cdots\right)$

24. $f(x) = x^2, \quad 0 < x < L$

Let $f_1(x) = x^2, \quad -L < x < L$

Then $b_n = 0$ for all n.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{1}{2L}\left[\frac{x^3}{3}\right]_{-L}^{L} = \frac{L^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\frac{n\pi}{L}x \, dx$$

$$= \left[\frac{x^2 \sin\frac{n\pi}{L}x}{n\pi}\right]_{-L}^{L} - \frac{L}{L}\int_{-L}^{L} \frac{2x}{n\pi}\sin\frac{n\pi}{L}x \, dx$$

$$= -\frac{2}{n\pi}\int_{-L}^{L} x \sin\frac{n\pi}{L}x \, dx$$

$$= \frac{2L}{n^2\pi^2}\left(\left[x\cos\frac{n\pi}{L}x\right]_{-L}^{L} - \int_{-L}^{L}\cos\frac{n\pi}{L}x \, dx\right)$$

$$= \frac{2L}{n^2\pi^2}\left(2L(-1)^n - \frac{L}{n\pi}\left[\sin\frac{n\pi}{L}x\right]_{-L}^{L}\right)$$

$$= \frac{4L^2}{n^2\pi^2}(-1)^n$$

$$\therefore f(x) = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\cos\frac{\pi x}{L} - \frac{1}{4}\cos\frac{2\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} - \cdots\right)$$

If you assume $f_1(x)$ is odd, i.e. $f_1(-x) = -x^2$, the method is similar. The answer is

$$\frac{2L^2}{\pi}\left[\left(1 - \frac{4}{\pi^2}\right)\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \left(\frac{1}{3} - \frac{4}{3^3\pi^2}\right)\sin\frac{3\pi x}{L} - \cdots\right)$$

11.4 Q9, $\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx$$

$$= \frac{1}{2\pi}\left[\frac{x e^{-inx}}{-in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-inx}}{in}dx$$

$$= \frac{(-1)^n}{-in} + 0$$

$$= i\frac{(-1)^n}{n} \quad \text{for } n \neq 0$$

$$\therefore f(x) = i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{inx}$$

11.9,

Q2, $F(iw) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(k-iw)x} dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(k-iw)x}}{k-iw} \right]_{-\infty}^{0}$

$= \frac{1}{\sqrt{2\pi}(k-iw)}$

Q6, $F(iw) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{\cancel{0}-1}^{1} x e^{-iwx} dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-iwx}}{-iw} \right]_{\cancel{0}-1}^{1} + \frac{1}{\sqrt{2\pi}} \int_{\cancel{0}-1}^{1} \frac{e^{-iwx}}{iw} dx$

$= \frac{-1}{i\sqrt{2\pi}w} (2\cos w) + \frac{1}{\sqrt{2\pi}w^2} \cdot (-2i \sin w)$

$= i\sqrt{\frac{2}{\pi}} \left(\frac{\cos w}{w} - \frac{\sin w}{w^2} \right)$

$= \frac{i}{w^2} \cdot \sqrt{\frac{2}{\pi}} (w\cos w - \sin w)$

Q10, $f(x) = x e^{-x}$

$f'(x) = -x e^{-x} + e^{-x}$

Apply Fourier Transform on both sides:

$iw F(iw) = -F(iw) + \frac{1}{\sqrt{2\pi}(1+iw)}$

$F(iw) = \frac{1}{\sqrt{2\pi}(1+iw)^2}$