1) a) If $r_1$ is double root, $f(r_1)=0$, $f'(r_1)=0$

Now, $L[xe^{rx}] = f'(r)e^{rx} + xf(r)e^{rx}$

Put $r=r_1$, $L[xe^{r_1x}]=0$, $\therefore xe^{r_1x}$ is a solution.

b) Suppose $r_1$ is a root of multiplicity $k$,

e.g. $f(r) = (r-r_1)^k(a_0r^{n-k} + \cdots)$ then

$f(r_1) = f^{(1)}(r_1) = \cdots = f^{(k-1)}(r_1) = 0$

We make a claim that $x^n e^{r_1x}$ is a solution to the $L[\,]$ for $0 \le n \le k-1$

For $n=0$, $L[e^{r_1x}] = f(r_1)e^{r_1x} = 0$

$n=1$, $L[xe^{r_1x}] = f'(r_1)e^{r_1x} + xf(r_1)e^{r_1x} = 0$

For $2 \le n \le k-1$,
$$L[x^n e^{rx}] = L\left[\frac{\partial}{\partial r}(x^{n-1}e^{rx})\right] = \frac{\partial}{\partial r} L[x^{n-1}e^{rx}]$$

Assume the result is true ~~for $2 \le p \le n-1$~~ for $2 \le p \le n-1$,

i.e. $L[x^p e^{r_1x}] = 0$ for $2 \le p \le n-1$,

Then for $p=n$, $L[x^n e^{r_1x}] = \frac{\partial}{\partial r} L[x^{n-1}e^{r_1x}] = 0$

So we can conclude that $x^n e^{r_1x}$ is a solution of $L[\,]$ for $r_1$ is a root of multiplicity $k$.

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2 a) By inspection, $y=x$ is a solution of $(**)$, $\therefore n=1$

b) Using reduction of order, $y=ux$, $y'=u'x+u$,

$y''=u''x+2u'$, $y'''=u'''x+3u''$

Put into $(**)$
$$(x^3\sin x)(u'''x+2u'') - (3x^2\sin x + x^3\cos x)(u''x+2u') + (6x\sin x + 2x^2\cos x)(u'x+u) - (6\sin x + 2x\cos x)ux = 0$$
$$(x^4\sin x)u''' - x^4\cos x\, u'' = 0$$
$$\sin x\, u''' - \cos x\, u'' = 0$$

cont. 21, Let $p = u''$, then $\sin x \cdot \frac{dp}{dx} - \cos x\, p = 0$

$$\int \frac{dp}{p} = \int \cot x \, dx$$

$$\ln p = \ln \sin x + C$$

$$p = C_0 \sin x$$

$$u' = C_0 \sin x + C_1$$

$$u = C_0 \sin x + C_1 x + C_2$$

$$\therefore y = C_0 x \sin x + C_1 x^2 + C_2 x$$

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Set 2.1, Q22, $(1-x^2)y'' - 2xy' + 2y = 0, \quad y_1 = x$

Let $y_2 = ux$, $y_2' = u'x + u$, $y_2'' = u''x + 2u'$

Sub into the equation gives

$$x(1-x^2)u'' + (2-4x^2)u' = 0$$

Let $p = u'$, $x(1-x^2)p' + 2(1-2x^2)p = 0$

$$\int \frac{dp}{p} = \int \frac{2(1-2x^2)}{x(1-x^2)} dx$$

$$= 2\int \frac{1}{x} - \frac{x}{1-x^2} dx$$

$$\ln p = 2\ln x + \ln(1-x^2) + C$$

$$p = C_1 x^2(1-x^2)$$

$$u' = C_1 x^2 (1-x^2)$$

$$u = C_1\left(\frac{x^3}{3} - \frac{x^5}{5}\right)$$

$$\therefore y_2 = \frac{x^4}{3} - \frac{x^6}{5}$$

$$y = C_0 x + C_2\left(\frac{x^4}{3} - \frac{x^6}{5}\right)$$

Set 2.4,

Q29, $20y''+4y'+y=0$, $y(0)=3.2$, $y'(0)=0$

Solve $20\lambda^2+4\lambda+1=0$

$$\lambda=\frac{-4\pm\sqrt{16-80}}{40}$$

$$\lambda=-\frac{1}{10}\pm\frac{1}{5}i$$

$$y=e^{-\frac{x}{10}}\left(A\sin\frac{x}{5}+B\cos\frac{x}{5}\right)$$

$y(0)=3.2$, $\Rightarrow B=3.2$

$$y'=e^{-\frac{x}{10}}\left(\frac{A}{5}\cos\frac{x}{5}-\frac{B}{5}\sin\frac{x}{5}\right)-\frac{1}{10}e^{-\frac{x}{10}}\left(A\sin\frac{x}{5}+B\cos\frac{x}{5}\right)$$

$y'(0)=\frac{A}{5}-\frac{B}{10}\Rightarrow A=1.6$

$$\therefore y=e^{-\frac{x}{10}}\left(1.6\sin\frac{x}{5}+3.2\cos\frac{x}{5}\right)$$

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Q30, $y''+2ky'+(k^2+w^2)y=0$, $y(0)=1$, $y'(0)=-k$

$$\lambda=\frac{-2k\pm\sqrt{4k^2-4(k^2+w^2)}}{2}=-k\pm wi$$

$$\therefore y=e^{-kx}(A\sin wx+B\cos wx)$$

$$y'=e^{-kx}(Aw\cos wx-Bw\sin wx)-ke^{-kx}(A\sin wx+B\cos wx)$$

$y(0)=1\Rightarrow B=1$

$y'(0)=-k=Aw-Bk$

$\therefore A=0$

$$y=e^{-kx}\cos wx$$

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Set 2.5,

Q.4, $x^2y''+3xy'+y=0$

Solve $m^2+2m+1=0$

$m=-1$

$$\therefore y=(C_1+C_2\ln x)x^{-1}$$

Set 2.7,

Q3, $y''-16y = 19.2\,e^{4x}+60e^{x}$

Solve $\lambda^2-16=0$

$\lambda = \pm 4$

$\therefore y_h = C_1e^{4x}+C_2e^{-4x}$

Let $y_p = Axe^{4x}+Be^{x}$

$y_p' = Ae^{4x}+4Axe^{4x}+Be^{x}$

$y_p'' = 4Ae^{4x}+16Axe^{4x}+4Ae^{4x}+Be^{x}$

$= 8Ae^{4x}+16Axe^{4x}+Be^{x}$

Sub into equation gives $8Ae^{4x}-15Be^{x} = 19.2e^{4x}+60e^{x}$

$\Rightarrow A=2.4,\ B=-4$

$\therefore y = C_1e^{4x}+C_2e^{-4x}+2.4xe^{4x}-4e^{x}$

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Q7, $y''+6y'+73y = 80e^{x}\cos 4x$

$\lambda^2+6\lambda+73=0$

$\lambda = \dfrac{-6\pm\sqrt{36-4(73)}}{2} = -3\pm 8i$

$y_h = e^{-3x}(C_1\sin 8x + C_2\cos 8x)$

$y_p = Ae^{x}\cos 4x + Be^{x}\sin 4x$

$y_p' = -4Ae^{x}\sin 4x + 4Be^{x}\cos 4x + Ae^{x}\cos 4x + Be^{x}\sin 4x$

$= (-4A+B)e^{x}\sin 4x + (A+4B)e^{x}\cos 4x$

$y_p'' = (-4A+B)e^{x}\sin 4x + (A+4B)e^{x}\cos 4x + (-16A+4B)e^{x}\cos 4x - (4A+16B)e^{x}\sin 4x$

$= (-8A-15B)e^{x}\sin 4x + (-15A+8B)e^{x}\cos 4x$

Sub into equation gives

$(-32A+64B)e^{x}\sin 4x + (64A+32B)e^{x}\cos 4x = 80e^{x}\cos 4x$

Upon solving, $A=1,\ B=\frac{1}{2}$

$y_p = e^{x}\cos 4x - \frac{1}{2}e^{x}\sin 4x$

$\therefore y = e^{-3x}(C_1\sin 8x + C_2\cos 8x) + e^{x}\cos 4x - \frac{1}{2}e^{x}\sin 4x$

Q16) $y'' - 3y' + 2.25y = 27(x^2 - x)$, $y(0) = 20$, $y'(0) = 30$

$\lambda^2 - 3\lambda + 2.25 = 0$

$\lambda = 1.5$

$y_h = (C_1 + C_2 x) e^{1.5x}$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

By compare coefficients give

$$\begin{cases} 2A - 3B + 2.25C = 0 \\ 2.25A = 27 \\ -6A + 2.25B = -27 \end{cases}$$

$\Rightarrow \{ A = 12,\ B = 20\ C = 16$

$y_p = 12x^2 + 20x + 16$

$y = (C_1 + C_2 x) e^{1.5x} + 12x^2 + 20x + 16$

$y(0) = 20 = C_1 + 16 \Rightarrow C_1 = 4$

$y' = (4 + C_2 x)(1.5 e^{1.5x}) + C_2 e^{1.5x} + 24x + 20$

$y'(0) = 6 + C_2 + 20 = 30$, $C_2 = 4$

$\therefore y = 4(1+x)e^{1.5x} + 12x^2 + 20x + 16$

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Q20) $y'' + 2y' + 10y = 17\sin x - 37\sin 3x$, $y(0) = 6.6$, $y'(0) = -2.2$

$\lambda^2 + 2\lambda + 10 = 0$

$\lambda = -1 \pm 3i$

$y_h = e^{-x}(C_1 \sin 3x + C_2 \cos 3x)$

let $y_p = A\sin x + B\cos x + C\sin 3x + D\cos 3x$

$y_p' = A\cos x - B\sin x + 3C\cos 3x - 3D\sin 3x$

$y_p'' = -A\sin x - B\cos x - 9C\sin 3x - 9D\cos 3x$

By substitute into the equation gives

$(9A - 2B)\sin x + (9B + 2A)\cos x + (C - 6D)\sin 3x + (D + 6C)\cos 3x$
$= 17\sin x - 37\sin 3x$

On solving, $A = 1.8$, $B = -0.4$, $C = -1$, $D = 6$

cont. 20),

$$y_p = 1.8\sin x - 0.4\cos x - \sin 3x + 6\cos 3x$$

$$y = e^{-x}(C_1\sin 3x + C_2\cos 3x) + 1.8\sin x - 0.4\cos x - \sin 3x + 6\cos 3x$$

$$y(0) = C_2 - 0.4 + 6 = 6.6 \Rightarrow C_2 = 1$$

$$y' = e^{-x}(3C_1\cos 3x - 3\sin 3x) + 1.8\cos x + 0.4\sin x - 3\cos 3x - 18\sin 3x$$
$$- e^{-x}(C_1\sin 3x + \cos 3x)$$

$$y'(0) = 3C_1 - 1 + 1.8 - 3 = -2.2 \Rightarrow C_1 = 0$$

$$\therefore y = e^{-x}\cos 3x + 1.8\sin x - 0.4\cos x - \sin 3x + 6\cos 3x$$

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2.10),

Q6), $x^2y'' - xy' + y = x\ln x$

solve $m^2 - 2m + 1 = 0$

$m = 1$

$\therefore y_1 = x,\ y_2 = x\ln x$

$y_h = (C_1 + C_2\ln x)x$

let $y_p = Ax(\ln x)^3$

$y_p' = A(\ln x)^3 + 3A(\ln x)^2$

$y_p'' = \frac{3A(\ln x)^2}{x} + \frac{6A\ln x}{x}$

$\therefore 3Ax(\ln x)^2 + 6Ax\ln x - xA(\ln x)^3 - 3Ax(\ln x)^2$
$+ Ax(\ln x)^3 = x\ln x$

$6Ax\ln x = x\ln x$

$A = \frac{1}{6}$

$\therefore y_p = \frac{x}{6}(\ln x)^3$

$y = (C_1 + C_2\ln x)x + \frac{x}{6}(\ln x)^3$