ERG 2011A Midterm solution

suggested by Vincent Wang

1. (a) can be solved by using E-C equation.

Assume the general solution y_h is in the form of x^m, then $y_h' = mx^{m-1}$, $y_h'' = m(m-1)x^{m-2}$ and $y_h''' = m(m-1)(m-2)x^{m-3}$. Substituting y_h, y_h', y_h'' and y_h''' into the original equation, we then have

$$x^m(m-1)(m-2)(m-3) = 0.$$

$$\Rightarrow \quad m = 1, 2, 3$$

So the homogeneous solution is $y_h = C_1 x + C_2 x^2 + C_3 x^3$

where C_1, C_2 and C_3 are constant.

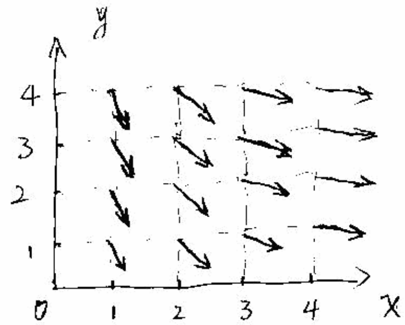
(b). According to the method of variation of parameters.

$$y_p(x) = y_1(x)\int \frac{W_1(x)}{W(x)} r(x)\,dx + y_2(x)\int \frac{W_2(x)}{W(x)} r(x)\,dx + y_3(x)\int \frac{W_3(x)}{W(x)} r(x)\,dx$$

Since $r(x) = 0$, $y_p(x)$ is alway equal to 0 whatever $W(x)$, $W_1(x)$, $W_2(x)$ and $W_3(x)$ are.

2,

(a)



(b) Separable method.

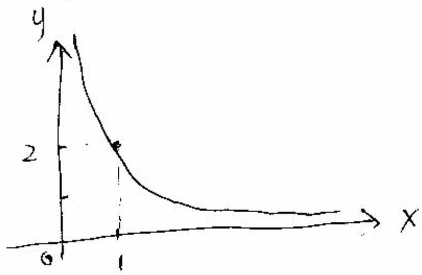
$$y' = \frac{-2}{x^2}$$

$$\frac{dy}{dx} = \frac{-2}{x^2}$$

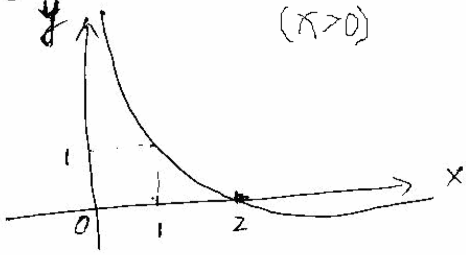
$$\int dy = -2\int \frac{1}{x^2}\,dx \Rightarrow y = \frac{2}{x} + C, \quad C \text{ is const.}$$

(c) sketch 2 solution curves to (b). Let's consider the two cases of C, C=0 and C=-1

when C=0. $y = \frac{2}{x}$ $(x>0)$



when C=-1. $y = \frac{2}{x} - 1$ $(x>0)$



3. Since the homogeneous solution are e^{2x} and xe^{2x}

we know $\lambda_1 = \lambda_2 = 2$.

which implies

$$y'' - 4y' + 4y = r(x) \quad \cdots\cdots \quad (1)$$

with general solution is $y = c_1 e^{2x} + c_2 x e^{2x} + (x + 1 + e^x)$.

So now, we need to find $r(x)$ ~~such~~ that (1) satisfies the general solution. Observe that

$$y' = (2c_1 + c_2) e^{2x} + 2c_2 x e^{2x} + 1 + e^x$$

$$y'' = (4c_1 + 4c_2) e^{2x} + 4c_2 x e^{2x} + e^x$$

Substituting y, y' and y'' into (1). we then have

$$r(x) = e^x + 4x,$$ with initial condition $y(0) = y'(0) = 2$

4. (a) $P(x,y) = 1$, $Q(x,y) = 2y(x+1)$

since $0 = \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} = 2y$.

The equation is not exact.

(b) An integrating factor is $R = \frac{1}{Q}(P_y - Q_x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{2y(x+1)}(-$

$$= -\frac{1}{x+1}$$

$$F(x) = e^{-\int R\,dx} = e^{-\int \frac{1}{x+1} dx} = \frac{1}{x+1}$$

(c) So now, $P'(x,y) = \frac{1}{x+1}$, $Q'(x,y) = 2y$ $\left(\frac{\partial P'}{\partial y} = \frac{\partial Q'}{\partial x}, \text{Exactness!}\right)$

$$\frac{du}{dx} = P' \Rightarrow u = \ln(x+1) + f(y)$$

$$\frac{du}{dy} = f'(y) = 2y \Rightarrow f(y) = y^2$$

So $u(x,y) = \ln(x+1) + y^2 = c$